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Convergence results for a semilinear problem and for a Stokes problem in a periodic geometry

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Abstract

In this paper, we study the asymptotic behavior of the solution of a semilinear problem and of a Stokes problem, with periodic data, when the size of the domain increases. In particular, we prove exponential convergence to the solution of the corresponding problem with periodic boundary conditions.

1 Introduction

At the end of the exploitation of an oil well, surface pumps are no longer efficient enough to extract the oil remaining in the reservoir. Instead of closing the well, oil companies might use well-pumps, introduced deep into the ground in order to maintain the production. Such pumps are composed of a succession of identical stages (typically 15-20 stages, but it could go up to 100 stages) arranged in series. The process optimization needs numerical simulation, but representing the whole pump numerically is impossible (for obvious calculation costs). Thus it is necessary to simplify the model, by representing only one stage. Therefore, a very natural question which arises is the following: does the flow in one standard stage of the pump looks like

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the one we would obtain by considering periodic boundary conditions at the entrance and exit of the stage. Of course, one cannot hope that this kind of result holds for the first and the last stages which are still influenced by the boundary conditions at the top and the bottom of the pump.

This kind of situation has already been considered in a series of papers by M. Chipot and Y. Xie, see [3], [5], [4], where the authors use variational techniques to prove the desired convergence. See also the book [2] by the first author. Their work is quite general and it definitely inspired us. Our own work differs to the following extent:

- we choose to work with the uniform (L^∞) norm, which might be more natural for the engineers,
- we use a simpler approach, just based on the maximum principle, which allows us to get a stronger convergence result (exponential convergence instead of polynomial convergence),
- in the elliptic case, we consider a semilinear problem instead of a linear one. Conversely, we are restricted to the Laplacian operator and to a simpler geometry (periodicity in only one dimension) than the one used by Chipot and Xie.
- motivated by the initial question, we also consider the Stokes problem in two dimensions (with particular boundary conditions: the so-called Navier slip boundary conditions).

Let us now be more precise in the statement of the problems and the results we get. We consider a domain $\Omega_k \subset \mathbb{R}^N$, composed of $2k$ copies of a bounded open set Ω translated in the last (N -th) space direction. For sake of simplicity, we assume that $\Omega \subset \mathbb{R}^{N-1} \times (0, L)$ for some positive L and that

$$\Omega_k = \bigcup_{i=-k}^{k-1} (iLe_N + \Omega).$$

In all the sequel, we assume that Ω_k is a domain of class $C^{1,1}$ for all k .

Let $f \in L^2_{loc}(\mathbb{R}^N)$ be a L -periodic function in the last space direction.

In section 2, we are interested in the following semilinear problem posed on Ω_k :

$$\begin{aligned} -\Delta u_k + g(x, u_k) &= f && \text{in } \Omega_k, \\ u_k &= 0 && \text{on } \partial\Omega_k, \end{aligned}$$

where the function g satisfies usual assumptions (see section 2 for more details) and let us denote by u_∞ the solution of the periodic problem posed

on Ω :

$$\begin{aligned} -\Delta u_\infty + g(x, u_\infty) &= f && \text{in } \Omega, \\ u_\infty &= 0 && \text{on } \Gamma, \\ u_\infty & \text{ } Le_N\text{-periodic.} \end{aligned}$$

where Γ is the lateral boundary of Ω . Then, we prove the convergence result:

Theorem 1.1. *For all $k_0 \in \mathbb{N}^*$, there exist two constants $K > 0$ and $\alpha > 0$, depending only on Ω_{k_0} , such that for all $k \geq k_0$,*

$$\|u_k - u_\infty\|_{L^\infty(\Omega_{k_0})} \leq K e^{-\alpha k}.$$

In section 3, we work in two dimensions ($N = 2$) and we consider the following Stokes problem:

$$\begin{aligned} -\Delta u_k + \nabla p_k &= f && \text{in } \Omega_k, \\ \nabla \cdot u_k &= 0 && \text{in } \Omega_k, \\ \text{curl } u_k &= 0 && \text{on } \partial\Omega_k, \\ u_k \cdot n &= 0 && \text{on } \partial\Omega_k. \end{aligned}$$

We are interested in proving the convergence of the solution to the previous problem, towards the solution of the following periodic Stokes problem:

$$\begin{aligned} -\Delta u_\infty + \nabla p_\infty &= f && \text{in } \Omega, \\ \nabla \cdot u_\infty &= 0 && \text{in } \Omega, \\ \text{curl } u_\infty &= 0 && \text{on } \Gamma, \\ u_\infty \cdot n &= 0 && \text{on } \Gamma, \\ u_\infty & \text{ } Le_2\text{-periodic.} \end{aligned}$$

We will prove that:

Theorem 1.2. *For all $k_0 \in \mathbb{N}^*$, there exist some positive constants K, K', α such that, for all $k \geq k_0$, we have*

$$\begin{aligned} \|u_k - u_\infty\|_{L^\infty(\Omega_{k_0})^2} &\leq K e^{-\alpha k}, \\ \|p_k - p_\infty\|_{L^2(\Omega_{k_0})} &\leq K' e^{-\alpha k}. \end{aligned}$$

At last, section 4 is devoted to some concluding remarks and possible extensions.

Notation:

x	$x = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N$
x'	$x' = (x_1, \dots, x_{N-1})$ where $x = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N$
Ω	bounded open set of \mathbb{R}^N such that $\Omega \subset \mathbb{R}^{N-1} \times [0, L]$, $L \in \mathbb{R}^{+*}$
Ω_k	union of $2k$ open set Ω , such that $\Omega_k \subset \mathbb{R}^{N-1} \times [-kL, kL]$
$\partial\Omega$	boundary of Ω
$\partial\Omega_k$	boundary of Ω_k
Σ_+	$\Sigma_+ = \partial\Omega \cap \{x_N = L\}$
Σ_-	$\Sigma_- = \partial\Omega \cap \{x_N = 0\}$
Σ	$\Sigma = \Sigma_+ \cup \Sigma_-$
Σ_{k+}	$\Sigma_{k+} = \partial\Omega_k \cap \{x_N = kL\}$
Σ_{k-}	$\Sigma_{k-} = \partial\Omega_k \cap \{x_N = -kL\}$
Σ_k	$\Sigma_k = \Sigma_{k+} \cup \Sigma_{k-}$
Γ	$\Gamma = \partial\Omega \setminus \Sigma$
Γ_k	$\Gamma_k = \partial\Omega_k \setminus \Sigma_k$

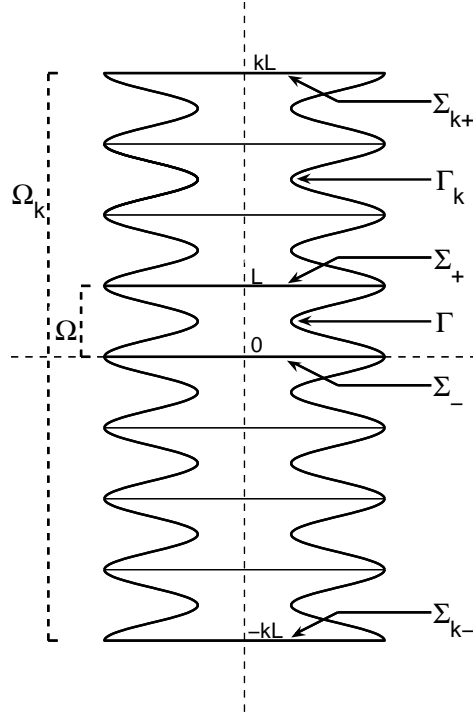


Figure 1: Periodic geometry as studied in this paper.

2 Semilinear problems in N -dimension

In this section, we consider a semilinear problem in a periodic geometry immersed in \mathbb{R}^N where $N \in \mathbb{N}$, $N \geq 2$. Let $f \in L^2_{loc}(\mathbb{R}^N)$ be a L -periodic function in the N -th space direction and let g be a function such that

$$x \mapsto g(x, \cdot)$$

is measurable and is a L -periodic function in the N -th space direction, and

$$s \mapsto g(\cdot, s) \text{ is continuous and increasing, } g(\cdot, 0) = 0. \quad (1)$$

Let us remark that the assumption $g(\cdot, 0) = 0$ is not a restriction since, otherwise, we could consider $g_1 = g - g(\cdot, 0)$ and transfer $g(\cdot, 0)$ in the right-hand side with f .

Moreover, let us assume that g satisfies the following property:

- if $N = 2$, there exist $r \in (1, \infty)$, $a_0 \in L^{\frac{r}{r-1}}(\Omega)$ and $b_0 \geq 0$, such that

$$|g(\cdot, s)| \leq a_0(\cdot) + b_0|s|^{r-1} \quad \text{in } \Omega; \quad (2)$$

- if $N \geq 3$, there exist $a_0 \in L^{\frac{2N}{N+2}}(\Omega)$ and $b_0 \geq 0$, such that

$$|g(\cdot, s)| \leq a_0(\cdot) + b_0|s|^{\frac{N+2}{N-2}} \quad \text{in } \Omega. \quad (3)$$

For all $k \in \mathbb{N}^*$, we denote by u_k the solution of the problem

$$\begin{aligned} -\Delta u_k + g(x, u_k) &= f \quad \text{in } \Omega_k, \\ u_k &= 0 \quad \text{on } \partial\Omega_k, \end{aligned} \quad (4)$$

and by u_∞ the solution of the problem

$$\begin{aligned} -\Delta u_\infty + g(x, u_\infty) &= f \quad \text{in } \Omega, \\ u_\infty &= 0 \quad \text{on } \Gamma, \\ u_\infty &\text{ } Le_N\text{-periodic.} \end{aligned} \quad (5)$$

We recall that with the above hypotheses on g , the systems (4) and (5) are well-posed. More precisely, we have that

Proposition 2.1. *Assume that Ω_k is of class $C^{1,1}$ and assume that g satisfies the above hypotheses. Then for any $f \in L^2_{loc}(\mathbb{R}^N)$ and for any $k \in \mathbb{N}$, there exists a unique solution $u_k \in H^1_0(\Omega_k)$ to the problem (4). Moreover u_k belongs to the Sobolev space $W^{2,q}(\Omega_k)$ for some $q > 1$ (depending on the dimension N).*

The same result holds for problem (5). The proof of this proposition is very classical. The standard way to prove existence and uniqueness consists in looking for minimizers of the functional

$$I(v) = \int_{\Omega_k} \frac{1}{2} |\nabla v(x)|^2 + G(x, v(x)) - f(x)v(x) \, dx,$$

where

$$G(x, u) = \int_0^u g(x, s) \, ds.$$

We easily check that I is strictly convex, coercive and lower semicontinuous on $H_0^1(\Omega_k)$ so that I attains its minimum only at u_k . Moreover, from assumption (2), (3) together with Lemma 17.1 and Corollary 17.2 of [11, pp. 64,65] we see that I is differentiable with

$$I'(v) = -\Delta v + g(x, v) - f.$$

Consequently, we have obtained the existence and uniqueness of a function $u_k \in V$ such that

$$-\Delta u_k + g(x, u_k) = f.$$

For the regularity of the solution, in dimension 2, since u_k belongs to L^r space, we have $g(x, u_k) \in L^{\frac{r}{r-1}}(\Omega_k)$ and then $u_k \in W^{2,q}(\Omega_k)$ with $q = \min\left(\frac{r}{r-1}, 2\right)$ by using the classical L^q regularity results (see e.g. [9, Theorem 9.15]). In the case $N \geq 3$, thanks to assumption (3), we have that $g(x, u_k) \in L^{2N/(N+2)}(\Omega_k)$. Therefore, using again the L^q regularity results (see e.g. [9, Theorem 9.15]), u_k belongs to the Sobolev space $W^{2,q}(\Omega_k)$ with $q = 2N/(N+2)$ and then $-\Delta u_k + g(x, u_k(x)) = f(x)$ holds a.e.

The main result of this section is the following theorem.

Theorem 2.1. *Assume that Ω_1 is of class $C^{1,1}$. Then for all $k_0 \in \mathbb{N}^*$, there exist two constants $K > 0$ and $\alpha > 0$, depending only on Ω_{k_0} , such that for all $k \geq k_0$,*

$$\|u_k - u_\infty\|_{L^\infty(\Omega_{k_0})} \leq K e^{-\alpha k}.$$

Proof. By periodicity assumption, the function u_∞ satisfies

$$-\Delta u_\infty + g(x, u_\infty) = f$$

in the whole domain Ω_k . Then, the function $v_k = u_k - u_\infty$ is solution of the following boundary value problem

$$\begin{aligned} -\Delta v_k + \left[\frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty} \right] v_k &= 0 && \text{in } \Omega_k, \\ v_k &= 0 && \text{on } \Gamma_k, \\ v_k &= -u_\infty && \text{on } \Sigma_k. \end{aligned} \tag{6}$$

Let B_{N-1} be a ball of \mathbb{R}^{N-1} such that $\Omega_k \subset\subset B_{N-1} \times (-kL, kL)$. We denote by λ_1 the first eigenvalue of the Laplacian with Dirichlet boundary conditions in B_{N-1} and we consider an associated eigenfunction φ_1 . We can assume that $\varphi_1 > 0$ in B_{N-1} . We define the function v in $B_{N-1} \times [-kL, kL]$ by

$$v(x) = M\varphi_1(x_1, \dots, x_{N-1}) \frac{\cosh(\sqrt{\lambda_1}x_N)}{\cosh(\sqrt{\lambda_1}kL)},$$

where M is a positive constant such that

$$M\varphi_1 \geq |u_\infty| \text{ on } \Sigma_k.$$

The function v satisfies the following system

$$\begin{aligned} -\Delta v &= 0 && \text{in } \Omega_k, \\ v &\geq 0 && \text{on } \Gamma_k, \\ v &\geq |u_\infty| && \text{on } \Sigma_k. \end{aligned} \tag{7}$$

By definition, we can notice that

$$v \geq 0 \text{ in } \Omega_k. \tag{8}$$

Using the fact that $s \mapsto g(\cdot, s)$ is an increasing function, we also have

$$\frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty} \geq 0 \text{ in } \Omega_k. \tag{9}$$

Therefore, from (6), (7), (8) and (9), we obtain

$$\begin{aligned} -\Delta(v - v_k) + \frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty}(v - v_k) \\ = \frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty}v \geq 0 \text{ in } \Omega_k. \end{aligned}$$

Consequently, the function $v - v_k$ satisfies

$$\begin{aligned} -\Delta(v - v_k) + \frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty}(v - v_k) &\geq 0 \text{ in } \Omega_k, \\ v - v_k &\geq 0 \text{ on } \partial\Omega_k. \end{aligned} \tag{10}$$

Likewise, the function $-v - v_k$ satisfies

$$\begin{aligned} -\Delta(-v - v_k) + \frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty}(-v - v_k) &\leq 0 \text{ in } \Omega_k, \\ -v - v_k &\leq 0 \text{ on } \partial\Omega_k. \end{aligned} \tag{11}$$

Then we apply the maximum principle [9, Theorem 8.1, p. 179] to the systems (10) and (11) and we get:

$$-v \leq v_k \leq v \text{ in } \Omega_k.$$

As a consequence, there exist two constants M' and K such that for all $k \geq k_0$,

$$\|u_k - u_\infty\|_{L^\infty(\Omega_{k_0})} \leq M' \frac{\cosh(\sqrt{\lambda_1} k_0 L)}{\cosh(\sqrt{\lambda_1} k L)} \leq K \exp(-\sqrt{\lambda_1} k L). \quad (12)$$

□

Remark 2.1. According to (12), the rate of decay (measuring the speed of convergence of u_k to u_∞) is given by $\alpha = \sqrt{\lambda_1} L$. If we want to get the best rate of decay, we have to choose a ball B_{N-1} as small as possible. Of course, the greater λ_1 , the greater the constant K appearing in (12). Besides, the optimal ball B_{N-1} , which would make the cylinder $B_{N-1} \times (-kL, kL)$ tangent to Ω_k , would let K going to $+\infty$.

3 The Stokes problem with Navier slip boundary conditions

In this section, we consider the Stokes problem in dimension 2 in space (i.e. $N = 2$). Let $f \in L^2_{loc}(\mathbb{R}^2)^2$ be a L -periodic function in the second space direction. For any $k \in \mathbb{N}$, we consider the solution (u_k, p_k) of the following Stokes problem in Ω_k (u_k is of course a vector-valued function):

$$\begin{aligned} -\Delta u_k + \nabla p_k &= f & \text{in } \Omega_k, \\ \nabla \cdot u_k &= 0 & \text{in } \Omega_k, \\ \text{curl } u_k &= 0 & \text{on } \partial\Omega_k, \\ u_k \cdot n &= 0 & \text{on } \partial\Omega_k. \end{aligned} \quad (13)$$

We also consider the solution (u_∞, p_∞) of the following periodic Stokes problem in Ω :

$$\begin{aligned} -\Delta u_\infty + \nabla p_\infty &= f & \text{in } \Omega, \\ \nabla \cdot u_\infty &= 0 & \text{in } \Omega, \\ \text{curl } u_\infty &= 0 & \text{on } \Gamma, \\ u_\infty \cdot n &= 0 & \text{on } \Gamma, \\ u_\infty & \text{Le}_2\text{-periodic.} \end{aligned} \quad (14)$$

We recall that for a vector function $u = (u_1, u_2)$, its curl is a scalar function defined by $\text{curl } u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$. Note that in the above systems, we have not

considered the classical no-slip condition but instead we have chosen the Navier slip boundary condition introduced by Navier in [12]

$$u \cdot n = 0 \quad \text{and} \quad \text{curl } u = 0 \quad \text{on } \Gamma.$$

The above condition can also be written under the form

$$u \cdot n = 0 \quad \text{and} \quad D(u)n \cdot \tau = 0 \quad \text{on } \Gamma,$$

where τ is the unit tangent vector field on Γ . This boundary condition corresponds to the physical hypothesis that the fluid slips at the border without friction. This slip boundary condition was derived from the Boltzmann equation in [6] and was already used by [8] and [7].

In the systems (13) and (14), the pressures p_k and p_∞ are unique up to a constant. In the sequel, we impose that

$$\int_{\Omega_k} p_k \, dx = 0, \quad \int_{\Omega} p_\infty \, dx = 0.$$

Let us also remark that, since $f \in L^2(\Omega)$, by classical ellipticity results the solution u_∞ belongs to $H^2(\Omega)$ and then to $L^\infty(\Omega)$. The same holds for u_k . We are now in position to state the main result of the paper.

Theorem 3.1. *For all $k_0 \in \mathbb{N}^*$, there exist three constants $K > 0$, $K' > 0$ and $\alpha > 0$, depending only on Ω_{k_0} , such that for all $k \geq k_0$,*

$$\|u_k - u_\infty\|_{L^\infty(\Omega_{k_0})^2} \leq K e^{-\alpha k} \quad \text{and} \quad \|u_k - u_\infty\|_{H_0^1(\Omega_{k_0})^2} \leq K e^{-\alpha k},$$

$$\|p_k - p_\infty\|_{L^2(\Omega_{k_0})} \leq K' e^{-\alpha k}.$$

Proof. According to [10], there exist two functions ψ_k and ψ_∞ such that

$$\begin{aligned} \text{curl } \psi_k &= \begin{pmatrix} \frac{\partial \psi_k}{\partial x_2} \\ -\frac{\partial \psi_k}{\partial x_1} \end{pmatrix} = u_k \quad \text{in } \Omega_k, \\ \psi_k &= 0 \quad \text{on } \partial\Omega_k, \end{aligned}$$

and

$$\begin{aligned} \text{curl } \psi_\infty &= u_\infty \quad \text{in } \Omega, \\ \psi_\infty &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Moreover, if we define

$$\omega_k = \text{curl } u_k \quad \text{and} \quad \omega_\infty = \text{curl } u_\infty,$$

then we have

$$\begin{aligned} -\Delta\psi_k &= \omega_k & \text{in } \Omega_k, \\ \psi_k &= 0 & \text{on } \partial\Omega_k, \end{aligned}$$

and

$$\begin{aligned} -\Delta\psi_\infty &= \omega_\infty & \text{in } \Omega_k, \\ \psi_\infty &= 0 & \text{on } \Gamma_k, \\ \psi_\infty & & Le_2\text{-periodic.} \end{aligned}$$

Besides, the function ω_k , respectively ω_∞ , satisfies the following boundary value problem:

$$\begin{aligned} -\Delta\omega_k &= \text{curl } f & \text{in } \Omega_k, \\ \omega_k &= 0 & \text{on } \partial\Omega_k, \end{aligned}$$

respectively,

$$\begin{aligned} -\Delta\omega_\infty &= \text{curl } f & \text{in } \Omega, \\ \omega_\infty &= 0 & \text{on } \Gamma, \\ \omega_\infty & & Le_2\text{-periodic.} \end{aligned}$$

The results of the previous section (with $g = 0$) apply here for these two systems. Nevertheless, we need to be more precise since we want to apply it to a right-hand side which is not periodic (but almost).

Let $R_a = (-a, a) \times [-kL, kL]$ be a rectangle which strictly contains $\overline{\Omega_k}$. In this rectangle, we consider the function

$$\phi : (x_1, x_2) \mapsto K_1 \cos(\pi x_1/2a) \frac{\cosh(\pi x_2/2a)}{\cosh(\pi kL/2a)}.$$

As we did in the previous section, it is possible to choose K_1 large enough so that, using the maximum principle, we get

$$|\omega_k - \omega_\infty| \leq \phi \text{ in } \Omega_k, \tag{15}$$

and then

$$\forall k_0 \in \mathbb{N}^*, \forall k \geq k_0, \|\omega_k - \omega_\infty\|_{L^\infty(\Omega_{k_0})} \leq K_1 \frac{\cosh(\pi k_0 L/2a)}{\cosh(\pi k L/2a)}.$$

We now want the same kind of inequality for ψ_k . We define $\phi_k = \psi_k - \psi_\infty$. Then

$$\begin{aligned} -\Delta\phi_k &= (\omega_k - \omega_\infty) & \text{in } \Omega_k, \\ \phi_k &= 0 & \text{on } \Gamma_k, \\ \phi_k &= -\psi_\infty & \text{on } \Sigma_k. \end{aligned} \tag{16}$$

We consider $\epsilon > 0$ such that

$$\overline{\Omega_k} \subset \subset (-a + \epsilon, a - \epsilon) \times [-kL, kL], \tag{17}$$

and we define the function

$$\phi_\epsilon : (x_1, x_2) \mapsto K_2 \cos\left(\frac{\pi x_1}{2(a-\epsilon)}\right) \frac{\cosh(\pi x_2/2a)}{\cosh(\pi kL/2a)}, \quad (18)$$

with a positive $K_2 \in \mathbb{R}$ to be specified further. We have

$$-\Delta\phi_\epsilon = \left(\frac{\pi}{2(a-\epsilon)}\right)^2 \phi_\epsilon - \left(\frac{\pi}{2a}\right)^2 \phi_\epsilon = \frac{\pi^2\epsilon(2a-\epsilon)}{4a^2(a-\epsilon)^2} \phi_\epsilon.$$

From (17) and (18), we can choose K_2 so that

$$\begin{aligned} -\Delta\phi_\epsilon &\geq \phi && \text{in } \Omega_k, \\ \phi_\epsilon &\geq |\psi_\infty| && \text{on } \partial\Omega_k. \end{aligned}$$

Combining (15) and (16) in the above system yields that

$$\begin{aligned} -\Delta(\phi_\epsilon - \phi_k) &\geq 0 && \text{in } \Omega_k, \\ \phi_\epsilon - \phi_k &\geq 0 && \text{on } \partial\Omega_k. \end{aligned}$$

In the same way, we also have that

$$\begin{aligned} -\Delta(-\phi_\epsilon - \phi_k) &\leq 0 && \text{in } \Omega_k, \\ -\phi_\epsilon - \phi_k &\leq 0 && \text{on } \partial\Omega_k. \end{aligned}$$

Using the maximum principle, it finally comes that $|\phi_k| \leq \phi_\epsilon$ i.e.

$$\forall k_0 \in \mathbb{N}^*, \forall k \geq k_0, \|\psi_k - \psi_\infty\|_{L^\infty(\Omega_{k_0})} \leq K'_2 \frac{\cosh(\pi k_0 L/2a)}{\cosh(\pi k L/2a)}, \quad (19)$$

for some positive constant K'_2 .

For $1 < \beta < \infty$, by applying a classical regularity result on elliptic equations (see [9, Theorem 9.14] for instance), we have that

$$\begin{aligned} \|\psi_k - \psi_\infty\|_{W^{2,\beta}(\Omega_{k_0})} &\leq c(\|\psi_k - \psi_\infty\|_{L^\beta(\Omega_{k_0})} + \|\omega_k - \omega_\infty\|_{L^\beta(\Omega_{k_0})}) \\ &\leq C \frac{\cosh(\pi k_0 L/2a)}{\cosh(\pi k L/2a)}. \end{aligned}$$

Since $u_k = \text{curl } \psi_k$ and $u_\infty = \text{curl } \psi_\infty$, the above inequality implies that

$$\|u_k - u_\infty\|_{W^{1,\beta}(\Omega_{k_0})^2} \leq C \frac{\cosh(\pi k_0 L/2a)}{\cosh(\pi k L/2a)} \leq K e^{-(\frac{\pi}{2a}L)k}. \quad (20)$$

For $\beta > 2$, we can use the Sobolev embedding

$$W^{1,\beta}(\Omega_{k_0}) \subset L^\infty(\Omega_{k_0}),$$

to get the L^∞ estimate for the velocity u_k .

We have now to estimate the difference $p_k - p_\infty$. To achieve this, we write the variational formulation corresponding to (13):

$$\forall v \in H_0^1(\Omega_{k_0})^2, \quad \mu \int_{\Omega_{k_0}} \nabla u_k : \nabla v \, dx - \int_{\Omega_{k_0}} p_k (\nabla \cdot v) \, dx = \int_{\Omega_{k_0}} f \cdot v \, dx \quad (21)$$

In the above equation, for any smooth $v : \Omega_k \rightarrow \mathbb{R}^2$, we have denoted by ∇v the matrix

$$(\nabla v)_{i,j} = \frac{\partial v_i}{\partial x_j}, \quad (i, j \in \{1, 2\})$$

and, for two matrices $A = (a_{i,j})_{i,j \in \{1,2\}}$ and $B = (b_{i,j})_{i,j \in \{1,2\}}$, we have denoted by $A : B$ the scalar product

$$A : B = \sum_{i,j \in \{1,2\}} a_{i,j} b_{i,j}.$$

Now, the periodic solution (u_∞, p_∞) defined on Ω can be naturally extended to Ω_k (see also [5, Lemma 2.4]) in such a way that it satisfies on Ω_k :

$$\forall v \in H_0^1(\Omega_{k_0})^2, \quad \mu \int_{\Omega_{k_0}} \nabla u_\infty : \nabla v \, dx - \int_{\Omega_{k_0}} p_\infty (\nabla \cdot v) \, dx = \int_{\Omega_{k_0}} f \cdot v \, dx. \quad (22)$$

Combining (21) and (22) yields that

$$\mu \int_{\Omega_{k_0}} (\nabla u_k - \nabla u_\infty) : \nabla v \, dx = \int_{\Omega_{k_0}} (p_k - p_\infty) \nabla \cdot v \, dx.$$

From (20) and the above equation, we get that

$$\int_{\Omega_{k_0}} (p_k - p_\infty) \nabla \cdot v \, dx \leq C e^{-(\frac{\pi}{2a}L)k} \|v\|_{H_0^1(\Omega_k)}$$

for any $v \in H_0^1(\Omega_k)^2$. This inequality and the inf-sup property (or LBB condition, see [10] or [1] for instance) imply that

$$\forall k_0 \in \mathbb{N}^*, \quad \forall k \geq k_0, \quad \|p_k - p_\infty\|_{L^2(\Omega_{k_0})} \leq C' e^{-(\frac{\pi}{2a}L)k}$$

with C' only depending on Ω_{k_0} . □

4 Conclusion

This paper does not completely answer to the question raised in the introduction. Indeed, the physical situation considered in the oil extraction would necessitate to extend the present work in three directions:

- consider the three-dimensional case,
- consider more general boundary conditions,
- consider the Navier-Stokes model instead of the Stokes problem.

It seems to us that each of these extensions is really challenging.

However, numerical simulations tend to confirm, in the Navier-Stokes case, the results proved for the Stokes equations. We use Fluent[®] to solve the Navier-Stokes equations in three dimensions for a gasoil flow in a pump. We study two cases: the first one is a six stages pump with velocity inlet and pressure outlet; the second one is a one stage pump with periodic conditions at the entrance and exit. We plot the velocity magnitude of the fluid in the pump, at a given radius. On Figure 2 the velocity profile appears to be the same in the fourth and fifth stages of the six stages pump. It indicates that the velocity profile in a stage seems also to converge in the 3D Navier-Stokes model.

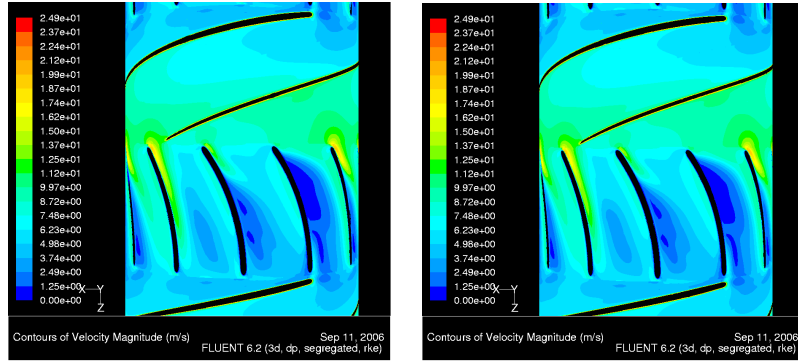


Figure 2: Velocity magnitude profile in the fourth (left) and fifth (right) stage of a six stages pump.

Figure 3 shows that the velocity profile in the fifth stage of a six stages pump and in a single stage with periodic conditions are very alike. It is a strong indication that our convergence result should also hold for the

Navier-Stokes problem (maybe with another rate of convergence). Though our simulation is limited to six stages, because of calculation costs, we could expect even better results with a bigger geometry.

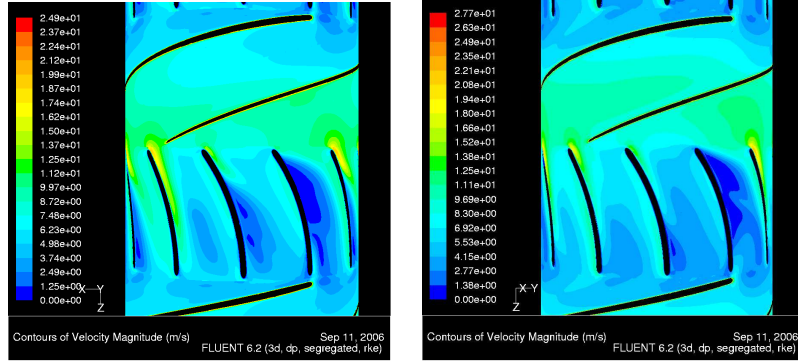


Figure 3: Velocity magnitude profile in the fifth stage of a six stages pump (left) and in a single stage with periodic conditions (right, the stage is displayed periodically).

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References

- [1] Franco Brezzi and Michel Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.
- [2] Michel Chipot. *ℓ goes to plus infinity*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2002.
- [3] Michel Chipot and Yitian Xie. On the asymptotic behaviour of elliptic problems with periodic data. *C. R. Math. Acad. Sci. Paris*, 339(7):477–482, 2004.

- [4] Michel Chipot and Yitian Xie. Asymptotic behavior of nonlinear parabolic problems with periodic data. In *Elliptic and parabolic problems*, volume 63 of *Progr. Nonlinear Differential Equations Appl.*, pages 147–156. Birkhäuser, Basel, 2005.
- [5] Michel Chipot and Yitian Xie. Elliptic problems with periodic data: an asymptotic analysis. *J. Math. Pures Appl. (9)*, 85(3):345–370, 2006.
- [6] François Coron. Derivation of slip boundary conditions for the Navier-Stokes system from the Boltzmann equation. *J. Statist. Phys.*, 54(3-4):829–857, 1989.
- [7] Jean-Michel Coron. On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions. *ESAIM Contrôle Optim. Calc. Var.*, 1:35–75 (electronic), 1995/96.
- [8] Giuseppe Geymonat and Enrique Sánchez-Palencia. On the vanishing viscosity limit for acoustic phenomena in a bounded region. *Arch. Rational Mech. Anal.*, 75(3):257–268, 1980/81.
- [9] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.
- [10] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.
- [11] Otared Kavian. *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, volume 13 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Paris, 1993.
- [12] C. L. M. H. Navier. Sur les lois du mouvement des fluides. *Mem. Acad. R. Sci. Inst. France*, 6:389–440, 1823.